Int. J. Solids Structures Vol. 15, pp. 127–136
© Pergamon Press Ltd, 1979 Printed in Great Britain

0020-7683/79/0201-0127/$02.00/0

FINITE ELEMENT STUDY OF THE FREE VIBRATION OF 3-D CABLE NETWORKS

M. L. GAMBHIR†
Thapar Engineering College, Patiala, Punjab, India

and

B. deV. BATCHELOR‡
Queen's University at Kingston, Ontario, Canada

(Received 22 March 1978; in revised form 11 July 1978)

Abstract—This paper briefly describes a finite element model previously developed for studying the free vibration characteristics of a single sagged cable hanging freely from two supports. This element, which allows elastic deformations, is used to determine the natural frequencies and normal modes of vibration of 3-D cable networks. A parametric study is made to predict the influence of various parameters, such as rise/span ratio of cables, initial pretension, cable rigidity, linear dimensions and surface curvature, on the natural frequencies and normal modes of vibration of 3-D cable nets. The results for various configurations are presented in the form of non-dimensional plots.

1. INTRODUCTION

Up to recently, the available literature on cable structures focussed principally on static linear behaviour. However, emphasis has now shifted to the important problem of dynamic instability including the phenomenon of flutter. The effects of flutter on cable nets have long been recognized, but the published literature has dealt in detail only with freely suspended cables; and only a few isolated design problems dealing with 3-D cable nets have been investigated. To enable evaluation of the response of the cable net to time-dependent forces such as aerodynamic and earthquake forces, it is necessary to determine certain basic characteristics of the network, including natural frequencies and normal modes of vibration.

Since the analysis of more practical configurations of cable nets is usually quite complex, investigators have resorted to simplified mathematical models for dynamic analysis of these systems. The most popular models are the continuous model[1–3] in which the cable net is modelled as an equivalent membrane; and the lumped mass model[1, 2, 4–8] in which all masses and loads are lumped at cable intersections (nodes). Investigators have found it necessary to impose quite restrictive limitations on the geometry of the system and the types of loading that can be accommodated in analysis.

The finite element technique has proved very effective for dealing with the dynamics of sagged cables[9, 10]. It enables quite detailed parametric studies to be done with a view to simplifying analysis as well as design. Leonard[11] has used a finite element to obtain the linear frequencies of a flat network problem previously analysed by Shore and Chaudhari[2]. Using the finite element method, Jensen[12] has studied the dynamic behaviour of suspension structures by idealizing the cable structure into a set of component string elements. He has also used plane constant strain elements to study membrane structures.

This paper reports an investigation using a finite element model to study the free vibration characteristics of 3-D cable networks. The results of an example problem are compared with those reported by Jensen[12]. A parametric study is then outlined to predict the influence of various parameters on the free vibration characteristics of 3-D cable nets; and the results are presented in the form of non-dimensional plots. It is surprising that in-plane motion, which may be critical for deep cable nets, seems not to have been considered in the technical literature. This aspect of vibration response is also discussed in the paper.

†Assistant Professor of Civil Engineering.
‡Professor of Civil Engineering.

2. OUTLINE OF THEORETICAL DEVELOPMENT OF FINITE ELEMENT

The numerical procedure used in the parametric study reported in this paper is based on representing the prestressed net as a series of finite length elements with linear elastic properties and possessing zero flexural rigidity. The elements are incapable of resisting compression and are connected at the nodes to form a 3-D network. Each node is assumed to have three degrees of freedom u, v and w, along the directions of the three orthogonal axes, and the extensional stiffness of the cable element is taken into account. The stiffness and consistent mass matrices associated with the equations of motion are derived using the principle of virtual work in its variational form. This formulation gives the stiffness and mass matrices of the element directly in terms of global coordinates and the displacements of the nodes, therefore, the method has the advantage of simplicity both in formulation and programming.

In an earlier investigation[9, 14] the equation of motion has been derived in the form:

$$[m]\{d\ddot{\Delta}\}+[k_G]\{d\Delta\}+[k_L]\{d\Delta\}+[k_{C1}]\{d\Delta\}+[k_{C2}]\{d\Delta\}=\{0\} \tag{1}$$

where, the vector $\{\Delta\}$ represents nodal coordinates and $\{d\Delta\}$ is the displacement vector. The matrix $[k_L]$ is the modified form of the conventional symmetric stiffness matrix associated with extension of the element; and $[k_G]$ is the geometric stiffness matrix. The matrices $[k_{C1}]$ and $[k_{C2}]$ account for nonlinear geometric effects and, respectively, contain the first and second order terms of the displacements $\{d\Delta\}$. These correction matrices take into account the changes in tension and in element length due to displacement.

For the typical element, ij, of length, l_{ij}, under a tension, T_{ij}:

$$\{\Delta\}=[x_i \quad y_i \quad z_i \quad x_j \quad y_j \quad z_j]^T \tag{2}$$

$$\{d\Delta\}=[u_i \quad v_i \quad w_i \quad u_j \quad v_j \quad w_j]^T \tag{3}$$

$$[k_G]=\int_0^1 T_{ij}l_{ij}[N']^T[N']\,d\xi \tag{4}$$

$$[k_L]=\int_0^1 (k_s-T_{ij})l_{ij}[[N']^T[N']\{\Delta\}\{\Delta\}^T[N']^T[N']]\,d\xi \tag{5}$$

$$\begin{aligned}[k_{C1}]=&\frac{1}{2}\int_0^1 (k_s-T_{ij})l_{ij}[2[N']^T[N']\{d\Delta\}\{\Delta\}^T[N']^T[N']+[N']^T[N']\{\Delta\}\{d\Delta\}^T[N']^T[N']]\,d\xi\\ &-\int_0^1 (k_s-T_{ij})l_{ij}[[N']^T[N']\{\Delta\}\{\Delta\}^T[N']^T[N']\{d\Delta\}\{\Delta\}^T[N']^T[N']]\,d\xi\end{aligned} \tag{6a}$$

$$\begin{aligned}[k_{C2}]=&\frac{1}{2}\int_0^1 (k_s-T_{ij})l_{ij}[[N']^T[N']\{d\Delta\}\{d\Delta\}^T[N']^T[N']]\,d\xi\\ &+\int_0^1 (k_s-T_{ij})l_{ij}[[N']^T[N']\{d\Delta\}\{\Delta\}^T[N']^T[N']\{d\Delta\}\{\Delta\}^T[N']^T[N']]\,d\xi\end{aligned} \tag{6b}$$

where $[N(x)]$ is the shape function relating the displacement field with the nodal coordinates; and $[N']$ is the ordinary differentiation of $[N]$ with respect to x. The term $k_s(=EA)$ represents the extensional rigidity of the element and $\xi(=x/l_{ij})$ varies from 0 to 1. In the case of a straight element:

$$[N]=[(1-\xi)[I] \quad \xi[I]] \tag{7}$$

and

$$[N']=\frac{1}{l_{ij}}[-[I] \quad [I]] \tag{8}$$

where $[I]$ is a (3×3) unit matrix and $[N']$ contains constant terms. Therefore,

$$[N']^T[N']=\frac{1}{l_{ij}^2}\begin{bmatrix} I & -I\\ -I & I\end{bmatrix}. \tag{9}$$

The consistent mass matrix is given by

$$[m] = \int_0^{l_{ij}} \left(\frac{\gamma}{g}\right)[N]^T[N]\,\mathrm{d}x = \frac{\gamma l_{ij}}{6g}\begin{bmatrix} 2[I] & [I] \\ [I] & 2[I] \end{bmatrix}. \tag{10}$$

The general equation (1) of motion is nonlinear. However, from the theoretical studies of freely suspended sagged cables[13] it has been observed that nonlinearity is slight. Therefore, in the parametric study presented in this paper, only small amplitude oscillations are considered. Accordingly, only the first three terms of eqn (1) are retained, giving:

$$[m]\{\mathrm{d}\ddot{\Delta}\} + [k_L]\{\mathrm{d}\Delta\} + [k_G]\{\mathrm{d}\Delta\} = \{0\}. \tag{11}$$

Substituting $[N']^T[N']$ from eqn (9) into eqns (4) and (5), the matrices $[k_G]$ and $[k_L]$ are given by:

$$[k_G] = \frac{T_{ij}}{l_{ij}}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \tag{12}$$

$$[k_L] = \frac{k_s - T_{ij}}{l_{ij}}\begin{bmatrix} G & -G \\ -G & G \end{bmatrix} \tag{13}$$

where

$$[G] = \begin{bmatrix} l^2 & lm & ln \\ ml & m^2 & mn \\ nl & nm & n^2 \end{bmatrix} \tag{14}$$

l, m and n being the direction cosines of element, ij, with respect to the global cartesian coordinate system and are given by:

$$l = (x_j - x_i)/l_{ij}, \qquad m = (y_j - y_i)/l_{ij} \qquad \text{and} \qquad n = (z_j - z_i)/l_{ij}. \tag{15}$$

A computer program based on the theory outlined above has been developed and tested on a variety of realistic problems. The program proves to be extremely efficient. Since all the computations were carried out on the computer in an integrated manner, it was possible in the study to vary tensions in the cables, boundary conditions and linear dimensions, etc. systematically.

The effects of various parameters on the modes of vibration and corresponding frequencies are studied. These parameters include rise/span ratio of cables, initial pretension, cable rigidity, linear dimension and curvature. In the basic configurations used in the study, the self weight has been taken as the initial load condition for calculating the equilibrium configurations about which vibrations are assumed to occur.

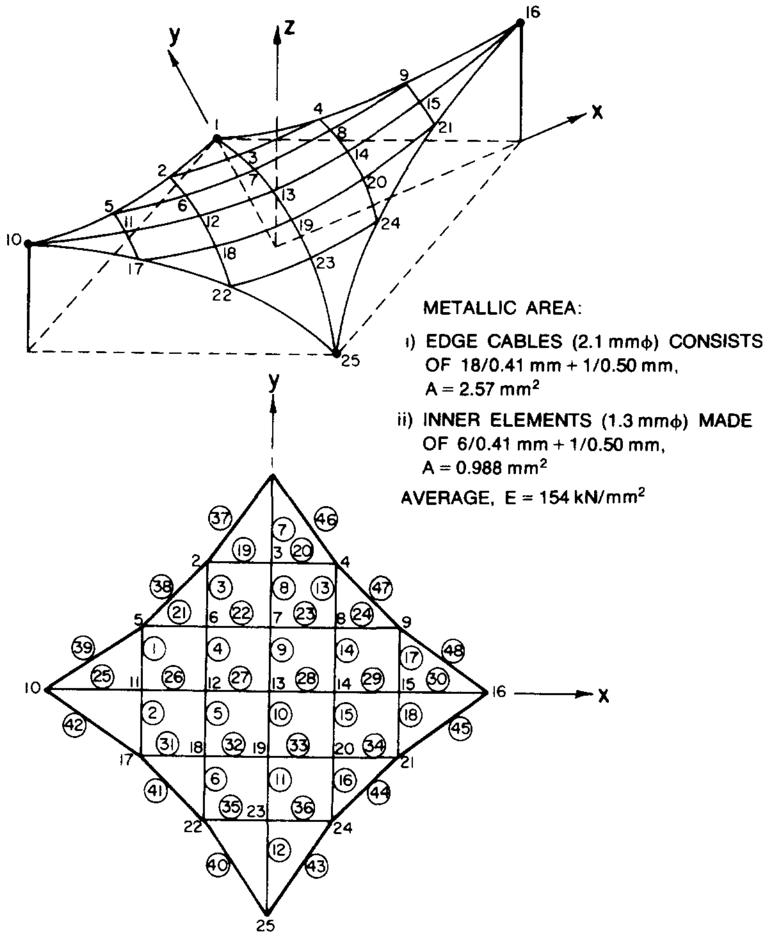
3. NUMERICAL RESULTS

The method has been illustrated by a numerical problem and the results compared with those obtained using the procedure described by an earlier investigator. The example deals with a 3-D cable network (Fig. 1), supported by edge cables. This network, which has been theoretically and experimentally investigated by Jensen[12], has a span of 2.1 m and a rise of 0.15 m, and carries a mass of 0.50 kg at each of the internal nodes.

In the previous investigation, the two lowest experimental frequencies were obtained by initiating vibrations by sudden release of an applied load. Jensen determined the higher modes of vibration by isolating them using an excitation technique. The frequencies reported by Jensen[12] are compared in Table 1 with those obtained in the authors' investigation.

4. PARAMETRIC STUDY

In the parametric study, a cable element is a segment of the cable between two cable intersections. The parameters involved in a cable net study are the mass per unit length,



PLAN OF CABLE NET

Fig. 1. Example problem of cable net with edge cables.

Table 1. Comparison of frequencies of 3-D network (Fig. 1)

Mode No.	Frequencies, in Hertz				
	Jensen [1]			Authors' Finite Element	
	Analytical	Experimental	Deviation, % †	Analytical	Deviation, % †
1	8.27	8.50	- 2.71	8.28	- 2.59
2	8.71	8.95	- 2.70	8.95	0.0
3	9.55	9.30	+ 2.70	9.73	+ 4.60
4	9.86	10.00	- 1.40	10.03	+ 0.30
5	10.07	10.70	- 5.90	10.51	- 1.78
6	10.55	10.70	- 1.40	10.66	- 0.37
7	12.11	11.80	+ 2.60	12.17	+ 3.14
8	12.34	12.15	+ 1.50	12.46	+ 2.55
9	-	12.95	-	-	-
10	13.04	-	-	13.48	-

† Deviations are computed with respect to experimental frequencies

$\rho(=\gamma/g)$; cable rigidity, $k_s(=EA)$; linear dimension, L; initial tension, T; rise/span ratio of the cable and the curvature of cable net surface.

In studying the effect of curvature on the free vibration characteristics, two types of 3-D cable nets have been analysed. The first type referred to as Type A, consists of a hyperbolic paraboloid network bounded by straight line generators and having cables parallel to these generators (Fig. 2a). The second type, referred to as Type B, consists of a hypar cable net having curved boundaries (Fig. 2b) and with cables having the same basic properties as those in Type A. The basic parameters of the system are as follows: linear dimension or span, $L_{y0} = L_{x0} = 90$ in. (2.3 m); rise, $h_x = h_y = 11.25$ in. (0.28 m); weight per unit length, $\gamma = 0.0026$ lb/in. (0.455 N/m); metallic area of cables, $A = 0.0125$ in^2 (8.0 mm^2); modulus of elasticity, $E = 10 \times 10^6$ psi (70 GPa); and horizontal component of tension, $T_{y0} = T_{x0} = 400$ lb (1780 N).

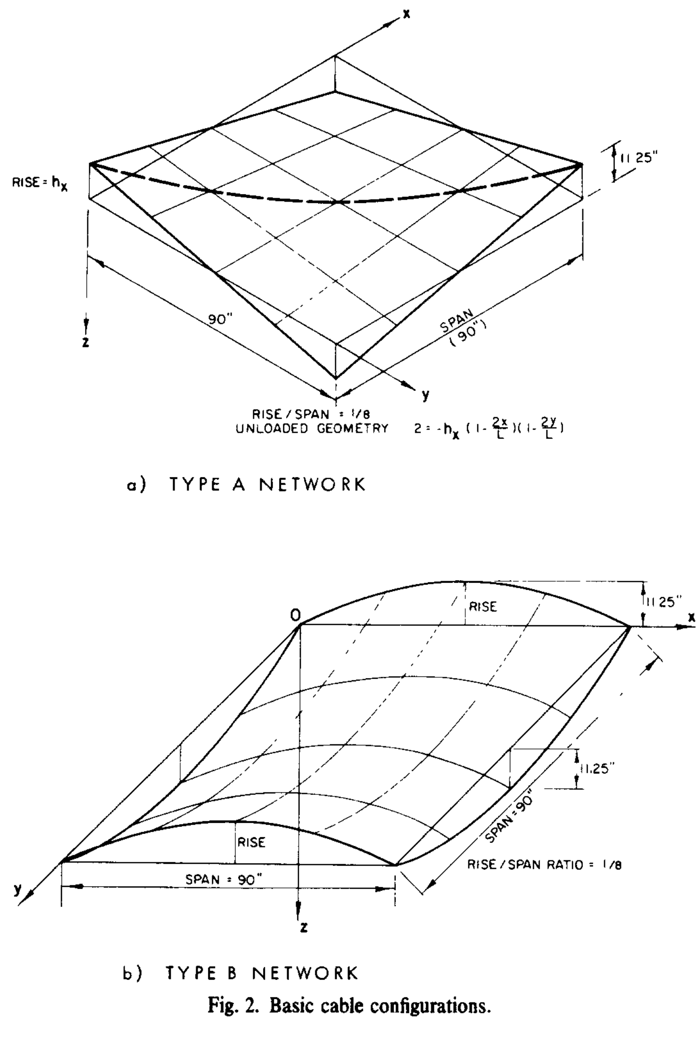


Fig. 2. Basic cable configurations.

5. EFFECTS OF PARAMETERS

5.1 *Cable rigidity and mass per unit length*

With other parameters being held constant, an increase in cable rigidity results from an increase in the cable cross-sectional area, A. It has been found in the study that the computed frequencies are unaltered by a variation of k_s or ρ. A careful inspection of the element mass and stiffness matrices (eqns 10, 12 and 13), reveals that the element stiffness matrices are multiplied either by $(k_s - T)$ or T, and that the elements of the mass matrix are multiplied by γ. Any change in area (and hence in γ) results in a corresponding change in k_s and T, therefore, the resulting frequencies are unaltered.

5.2 *Linear dimension*

The linear dimensions of the basic structure are varied in steps of 45 in. (1.14 m) over the range 45 in. (1.14 m) to 225 in. (5.72 m). The resulting frequencies for the two types of nets are plotted in Figs. 3 and 4, which show that a hyperbolic relationship exists between frequency and linear dimensions. The plot for the Type B network is shown in Fig. 4 on a logarithmic scale to have a slope of unity. This indicates that frequencies are inversely proportional to span or linear dimension.

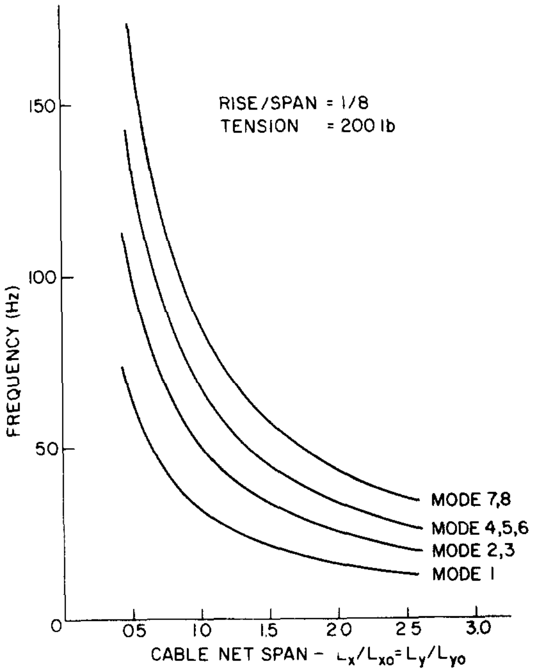


Fig. 3. Variation of frequency with linear dimension (Type A network).

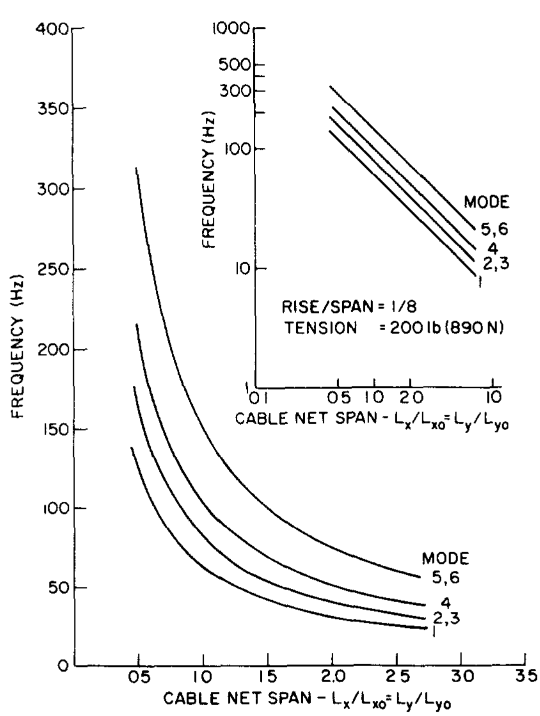


Fig. 4. Variation of frequency with linear dimension (Type B network).

5.3 *Initial tension*

The rise/span ratio of both cable nets is held constant at 1/8, while the pretension in each cable is varied from 200 lb (890 N) to 1000 lb (4.45 kN). The calculated frequencies for the first six modes are plotted in Figs. 5 and 6. When the results for the Type A network are plotted on a logarithmic scale, a slope of 0.5 is obtained, indicating that the frequencies are proportional to $\sqrt{T}$, i.e. frequencies correspond to taut string frequencies. A similar relationship holds for Type B network, except that at high tension the relationship at higher frequencies is slightly nonlinear. The magnitude of pretension has been found to have no influence on mode shapes for Type A network. However, in the case of Type B network, the mode shapes 1–3 are independent of pretension, while modes 4 and 6 interchange their forms at a value of $T_x/T_{x0} = T_y/T_{y0} \approx 2.7$.

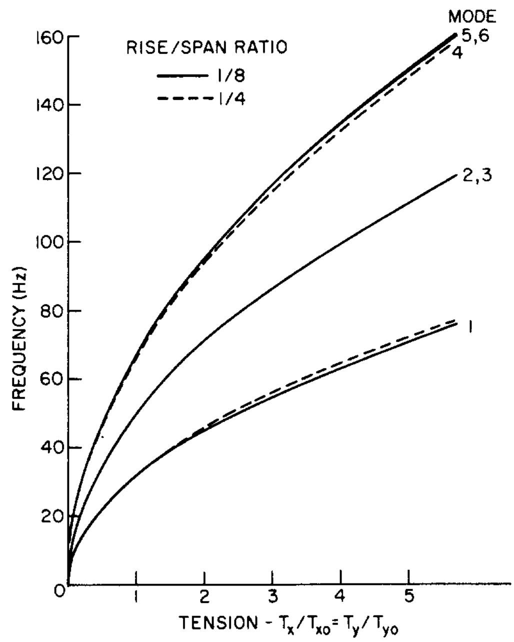


Fig. 5. Variation of frequency with pretension (Type A network).

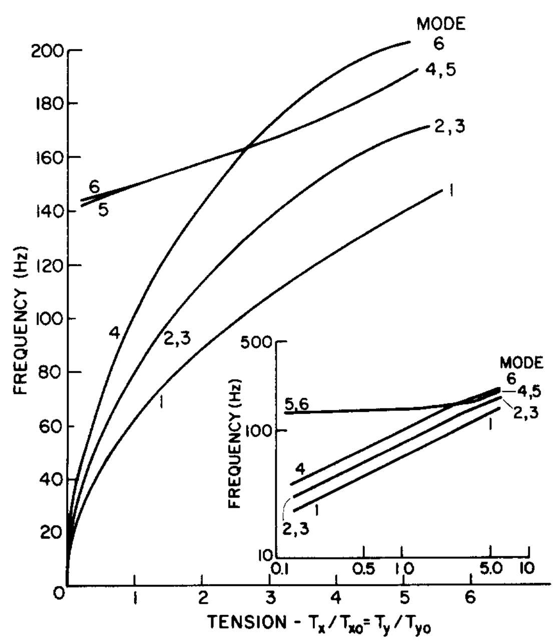


Fig. 6. Variation of frequency with pretension (Type B network).

5.4 *Rise/span ratio*

The rise/span ratio for the X and Y cables ($h_x/L_{x0} = h_y/L_{y0}$) has been varied from 0 to 5/16 and the plots of natural frequency vs rise/span ratio are shown in Figs. 7 and 8. The study indicates that the transverse modes yield considerably lower frequencies than do the in-plane modes. Therefore, in most practical problems, only transverse modes need be considered.

It can be seen (Fig. 7) that as the rise/span ratio of the Type A network is increased, the mode shapes change. For rise/span = 0, the 4th, 5th and 6th mode shapes are identical. As the rise/span ratio is increased the 4th mode deviates from 5th and 6th modes over the entire range of rise/span ratios studied; the 2nd mode is identical to the 3rd, and the 5th mode is identical to the 6th. However, the first natural frequency increases with rise/span ratio whereas all other frequencies decrease.

For the Type B hypar network, the variation of natural frequencies with rise/span ratio is

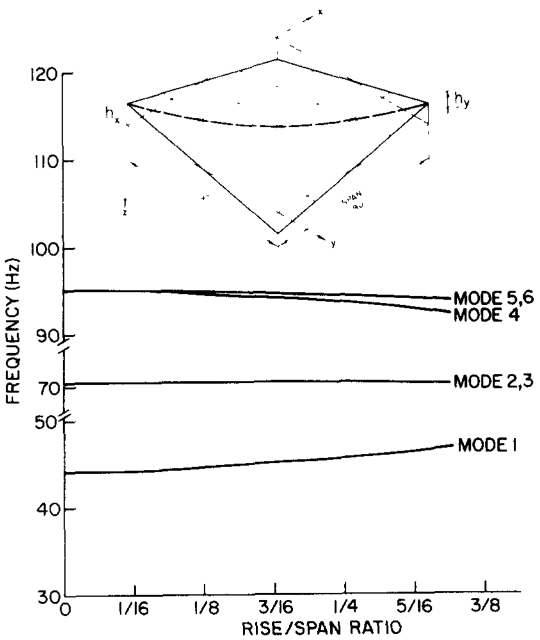


Fig. 7. Natural frequency vs rise/span ratio (Type A network).

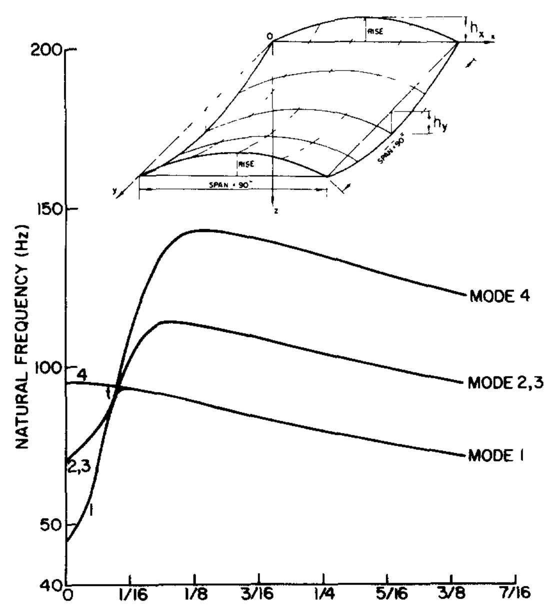


Fig. 8. Natural frequency vs rise/span ratio (Type B network)

given in Fig. 8 for the first four modes. In contrast to the Type A network formed by straight line generators, the Type B network shows large variations in frequency with rise/span ratio, i.e. curvature has a pronounced effect on frequencies. At the common intersection point obtained by connecting modes 1 and 4, these modes (1 and 4) interchange forms. This intersection point may be called a transition point, which shifts towards the right with an increase in tension, T, of the cables. Similar transition or cross-over points have been observed in the study of single sagged cables [9, 10]. With tension values of 200 lb (890 N), 400 lb (1.78 kN), 600 lb (2.67 kN) and 800 lb (3.56 kN), the corresponding transition points are obtained at rise/span values of approximately 1/32, 1/20.5, 1/16.5 and 1/14.5, respectively. The frequencies corresponding to the 2nd and 3rd modes are coincident, although there is a change in form beyond rise/span values corresponding to transition points. It is interesting to note that the transition point corresponds to the highest frequency for mode 1 for any rise/span ratio. At this transition point the first mode with no nodal lines (half wave forms parallel to the X and Y axes) changes to a mode with one nodal line (i.e. full wave form parallel to X and Y axes).

5.5 *Surface curvature*

The surface curvature $[(8h_x/L_x^2) = (8h_x/L_x) \cdot (1/L_x)]$ of the cable net surface can be varied either by a change in rise/span ratio (h_x/L_x) with linear dimension (L_x) held constant, or by varying the linear dimension while rise/span ratio is held constant. These two cases have been studied in Sections 5.2 and 5.4, respectively, where it has been shown that the frequencies are inversely proportional to linear dimension, therefore, it may be inferred that the frequencies are linearly related to surface curvature. Since the Type A network consists of straight cables having zero curvature, the frequencies are affected by surface curvature to a much lesser degree than those of the Type B network. No change in mode shapes is observed as a result of variation in linear dimension, so that mode shapes will also be independent of surface curvature.

6. CONCLUSIONS

The following conclusions are made from the finite element parametric study reported in the paper:

(1) Natural frequencies of cable networks are independent of the axial rigidity of an individual cable and are primarily dependent on tension and secondarily on rise/span ratio.

(2) If the linear dimensions of the cable net system are varied while all other basic parameters are held unchanged, then the frequencies of the network are inversely proportional to the linear dimension. However, the mode shapes are not influenced by the change in linear dimension.

(3) Modal configurations are independent of the magnitude of the pretension, but the squares of the first few frequencies increase linearly with pretension. For higher modes at high prestress value, this relationship is slightly nonlinear.

(4) For a rise/span ratio of zero, i.e. for flat networks, the frequencies are closely related to those of a taut cable of the system. For 3-D cable networks having moderate curvatures the transverse modes yield considerably lower frequencies than do the in-plane modes. Unless the network is excessively deep, only transverse modes need by considered in analysis. In the case of 3-D hypar networks bounded by straight line generators and incorporating straight cables, the behaviour is similar to that of a single taut cable in the system. In general, as the rise/span ratio is increased, there occurs a transition in modal configurations requiring reordering of the mode shapes corresponding to the ordered array of frequencies. The value of rise/span ratio corresponding to the transition point yields maximum frequency for the lowest mode.

(5) The squares of frequencies increase linearly with curvature of the network, whereas mode shapes are independent of curvature.

It can be seen, therefore, that the finite element technique proves to be effective in giving an insight into the general characteristics of cable structures. The procedure effectively reveals the general order of magnitude and influence of various parameters on the spectrum of frequencies of a cable net.

Acknowledgements—The authors acknowledge the support of the Canadian Commonwealth Scholarship and Fellowship Committee and National Research Council of Canada.

REFERENCES

1. K. G. Sangster, Dynamic Response of Cable Networks. Ph.D. Thesis, Queen's University at Kingston, Ontario, Canada (1974).
2. S. Shore and B. Chaudhari, Free Vibration of Cable Networks Utilizing Analogous Membrane. *Proc. Int. Assoc. Bridges and Structural Engng, 9th Cong.*, Amsterdam, 445–451 (1972).
3. A. I. Soler and H. Afshari, On the analysis of cable network vibrations using Galerkin method. *J. Appl. Mech., Proc ASME* **37**, 606–611 (1970).
4. M. L. Gambhir and B. deV. Batchelor, Free vibrations of taut cables and plane cable nets. *J. Struct. Div., Proc. ASCE* **103**, 2264–2268 (1977).
5. W. C. Knudson, Response of Cable-Net Structures under Dynamic Loads. *Proc. IASS Pacific Symp. Part II on Tension Structures and Space Frames.* Tokyo, 365–376 (1971).
6. D. Modjtahedi and S. Shore, The nonlinear steady state response of flexible cable networks. *Proc. IASS Pacific Symp. Part II on Tension Structures and Space Franes.* Tokyo, 353–364 (1971).
7. V. Svars and L. C. Schmidt, Vibration of Pretensioned Cable Nets. *4th Australasian Conf. on Mech. of Structures and Materials.* University of Queensland, Brisbane, 285–290 (1973).
8. S. S. Tezcan, S. Cherry and B. C. Mahapatra, Dynamic Analysis of Cable Strucutres. *Proc. IASS Pacific Symp. Part II on Tension Structures and Space Frames.* Tokyo, 401–410 (1971).
9. B. deV. Batchelor, M. L. Gambhir, Dynamics of Sagged Cables by Finite Element Method. *Proc. Symp. Applications of Computer Methods in Engng.* Los Angeles, California, 681–690 (1977).
10. M. L. Gambhir and B. deV. Batchelor, Parametric study of free vibration of sagged cables. *J. Comput. Structures.* Vol. 8, pp. 641–648 (1978).
11. J. W. Leonard, Incremental response of 3-D cable networks. *J. Engng Mech. Div. Proc. ASCE* **99**, 621–629 (1973).
12. J. J. Jensen, Eine Statische und Dynamische Untersuchung der, Seil-und Membrantragwerke. *Rep. No.* 70-1, Div. Struct. Mech., Norwegian Institute of Tech., University of Trondheim, Norway (1970).
13. M. L. Gambhir and B. deV. Batchelor, Discussion on Free Vibration of Cables in Three Dimension by W. M. Henghold, J. J. Russell and J. D. Morgan, *J. Struct. Div., Proc. ASCE* **104**, 613–615 (1978).
14. M. L. Gambhir and B. deV. Batchelor, A finite element for 3-D prestressed cable nets. *Int. J. Num. Meth. Engng* **11**, 1699–1718 (1977).
15. S. Shore and G. N. Bathish, *Membrane Analysis of Cable Roofs in Space Structures* (Edited by R. M. Davies), pp. 890–906. Blackwell, Oxford (1967).